Derivation of Black-Scholes formula

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1 Goal:

To show that under the model

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

the no-arbitrage price V_0 of a European call with strike K and expiration time T satisfies

$$V_0 = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{(r + \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}} \\ d_2 &= \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}} \end{aligned}$$

2 Derivation of Black-Scholes formula

1. By the risk neutral pricing formula:

$$\begin{aligned} V_0 &= E(e^{-rT}V_T) = E(e^{-rT}(S_T - K)^+) = E\Big[e^{-rT}\big(S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma B_T} - K\big)^+\Big] \\ &= E\Big[e^{-rT}\big(S_0 e^X - K\big)^+\Big], \end{aligned}$$

where X has distribution $N((r - \frac{1}{2}\sigma^2)T, \sigma^2 T)$. Note that $S_0 e^X - K \geq 0$ iff $X \geq \log(\frac{K}{S_0})$. Therefore

$$\begin{aligned} V_0 &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT}(S_0 e^x - K)\phi_X(x)dx \\ &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 e^x \phi_X(x)dx - \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} K\phi_X(x)dx \\ &:= A - B, \end{aligned}$$

where $\phi_X(x) := \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2 T}\big), \mu := (r - \frac{1}{2}\sigma^2)T$ is the density of X. We will compute A and B separately.

2.

$$\begin{aligned} B &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} K \phi_X(x) dx = e^{-rT} K P(X > \log(\frac{K}{S_0})) \\ &= e^{-rT} K P(Z > \frac{\log(\frac{K}{S_0}) - (r - \frac{1}{2}\sigma^2)T}{\sigma T}) \\ &= e^{-rT} K N(d_2), \end{aligned}$$

where

$$d_2 := \frac{(r - \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}.$$

3.

$$\begin{aligned} A &= \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 e^x \phi_X(x) dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 \exp\big(x - \frac{(x-\mu)^2}{2\sigma^2 T}\big) dx. \end{aligned}$$

Clearly,

$$x - \frac{(x-\mu)^2}{2\sigma^2 T} = \frac{2\sigma^2 T x - (x-\mu)^2}{2\sigma^2 T}.$$

We complete the square in the numerator of the above fraction:

$$\begin{aligned} 2\sigma^2 T x - (x-\mu)^2 &= -x^2 + 2(\sigma^2 T + \mu)x - \mu^2 = -\big(x - (\sigma^2 T + \mu)\big)^2 + (\sigma^2 T + \mu)^2 - \mu^2 \\ &= -(x - (\sigma^2 T + \mu))^2 + \sigma^4 T^2 + 2\mu\sigma^2 T. \end{aligned}$$

Thus

$$\begin{aligned} x - \frac{(x-\mu)^2}{2\sigma^2 T} &= \frac{-(x - (\sigma^2 T + \mu))^2 + \sigma^4 T^2 + 2\mu\sigma^2 T}{2\sigma^2 T} \\ &= -\frac{(x - (\sigma^2 T + \mu))^2}{2\sigma^2 T} + \frac{1}{2}\sigma^2 T + \mu. \end{aligned}$$

Also note that

$$\frac{1}{2}\sigma^2 T + \mu = \frac{1}{2}\sigma^2 T + (r - \frac{1}{2}\sigma^2)T = rT.$$

and

$$\sigma^2 T + \mu = \sigma^2 T + (r - \frac{1}{2}\sigma^2)T = (r + \frac{1}{2}\sigma^2)T.$$

Therefore

$$\begin{aligned}
A &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} e^{-rT} S_0 \exp\big(x - \frac{(x-\mu)^2}{2\sigma^2 T}\big) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\log(\frac{K}{S_0})}^{\infty} S_0 \exp\big(-\frac{(x-(r+\frac{1}{2}\sigma^2)T)^2}{2\sigma^2 T}\big) dx \\
&= S_0 P\big(\tilde{X} \geq \log(\frac{K}{S_0})\big),
\end{aligned}$$

where $\tilde{X}$ has distribution $N((r+\frac{1}{2}\sigma^2)T, \sigma^2 T)$. Thus

$$\begin{aligned}
A &= S_0 P(Z \geq \frac{\log(\frac{K}{S_0}) - (r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}) \\
&= S_0 N(d_1),
\end{aligned}$$

where $d_1 = \frac{(r+\frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}$. This finishes the derivation of Black-Scholes formula.